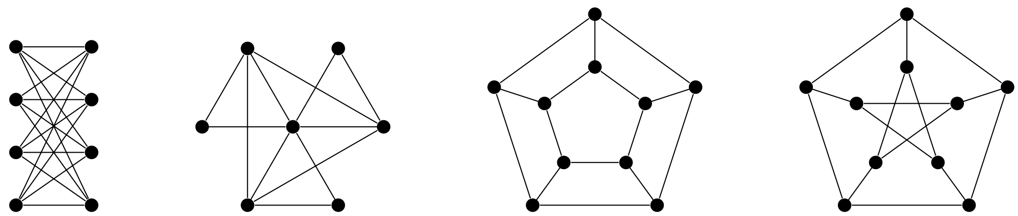
Chapter 10 Hamilton Cycles

A graph G on n vertices is **Hamiltonian** if it contains a cycle of length n. The cycle is called **Hamilton cycle** (or Hamiltonian cycle). (Imagine you want to visit every vertex of a graph once. You don't care about edges.) A path containing all vertices is called **Hamilton path** (or Hamiltonian path).

1: Decide for the following graphs if they are Hamiltonian, have Hamilton path or nothing.

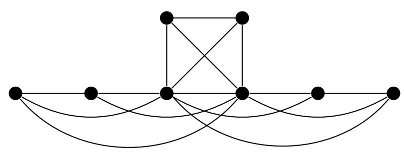


Solution: Both, Path, Both, Path

2: Is there a Hamiltonian graph that is not 2-connected? (i.e, connectivity is 1)

Solution: No.

3: Is the following graph Hamiltonian? Why?



Solution: The graph contains a cut S of size two such that the $G - S$ has three connected components.

Recall that $k(G)$ denotes the number of connected components of a graph G.

Lemma If G is a Hamiltonian graph, then for every nonempty proper set S of vertices of G,

$$k(G - S) \leq |S|.$$

4: Prove the Lemma

Solution: Let S be a cut and $G_1, \ldots, G_k$ connected components of $G - S$. Consider the Hamilton cycle. Every component has at least two neighbors in S and every vertex in S has degree 2 on the cycle. Hence $|S| \geq k$.

Theorem 1 Let G be a graph of order $n \geq 3$. If

$$\deg u + \deg v \geq n$$

for each pair u, v of nonadjacent vertices of G, then G is Hamiltonian.

Proof Fix n. Let G be a counterexample maximizing the number of edges (why can we take it?). Notice $G \neq K_n$ so G has u, v nonadjacent vertices. By maximality of the number of edges, $G + uv$ contains a Hamilton cycle C and C contains edge uv. Then $C - uv$ is a Hamilton path P with endpoints u and v.

5: How to finish the proof? How to use neighbors of u and v and P to find a different Hamilton cycle? Find construction or contradiction with $\deg u + \deg v \geq n$.

Solution: Idea: Let $u = u_1, u_2, \ldots, u_n = v$ be the Hamiltonian path. Observe that if uu_j is an edge, then vu_{j-1} is not an edge. Hence for every edge from u, there is a non-edge from v. Let $k = \deg u$. Observe that $\deg v \leq (n-1) - \deg u$. Hence $\deg v + \deg u \leq n - 1$ which is a contradiction.

6: Show that Theorem 1 is sharp by finding a graph G where for every pair of non-adjacent vertices u and v satisfy $deg(u) + deg(v) \geq n - 1$ but G is not Hamiltonian.

Solution: Two complete graphs sharing a vertex will do it.

Theorem 2 Let u and v be nonadjacent vertices in a graph G of order n such that $\deg u + \deg v \geq n$. Then $G + uv$ is Hamiltonian if and only if G is Hamiltonian.

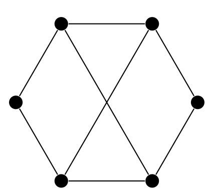
7: Prove Theorem 2 (see proof of Theorem 1)

Solution: If addition of the edge uv creates a Hamilton cycle, we get exactly the same piece of proof as in Theorem 1.

The **closure** $C(G)$ of a graph G of order n is obtained from G by adding edges between pairs of vertices u and v where $\deg u + \deg v \geq n$.

Theorem 2 implies that G is Hamiltonian iff $C(G)$ is Hamiltonian.

8: Find closure of the following graph



Theorem 3 Let G be a graph of order $n \geq 3$. If for every integer j with $1 \leq j < \frac{n}{2}$, the number of vertices of G with degree at most j is less than j, then G is Hamiltonian.

Proof: We show that the closure of G is a complete graph. Suppose for contradiction that it is not. Let u and w be two non-adjacent vertices of the closure $C(G)$ such that $\deg_{C(G)} u + \deg_{C(G)} w$ is as large as possible. Let $k = \deg_{C(G)} u \leq \deg_{C(G)} w$.

9: Finish the proof. Show $k < \frac{n}{2}$. Consider W the set of vertices non-adjacent to w and look at $|W|$.

Solution: In W have degree $\leq k$. So $|W| \leq k-1$. Hence $\deg_{C(G)} w \geq n-1$ and $\deg_{C(G)} u + \deg_{C(G)} w \geq n$.

Recall $\alpha(G)$ is the size of the largest independent set and $\kappa(G)$ is the connectivity of G.
Theorem 4 Let G be a graph of order $n \geq 3$. If $\alpha(G) \leq \kappa(G)$, then G has a Hamilton cycle.

Let $k := \kappa(G)$. Let $C := v_1, v_2, \ldots, v_\ell$ be the longest cycle in G. If C is Hamilton cycle, we are done. Suppose for contradiction that C is not a Hamilton cycle. Since C is not a Hamilton cycle, there exists $u \in V(G) \setminus V(C)$.

By Menger's theorem, there exists paths P_i that are internally disjoint with C and P_i is a v_i-u path and these paths are disjoint except for u. Let I be the set of indices I, where these paths exist. Take the largest I.

10: Show that $|I| \geq \min\{k, |V(C)|\}$.

Solution: This is a consequence of Menger's theorem. We are looking for C-u paths, there must be at least k of them, unless $k > |V(C)|$ and then there are $|V(C)|$ of them.

11: Show that if $i \in I$, then $i+1 \notin I$. In other words, there are no internally disjoints paths v_i-u and v_{i+1}-u.

Solution:

12: Show that if $i, j \in I$, then $v_{i+1}v_{j+1}$ is not an edge of G.

Solution:

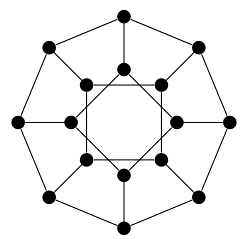
13: Find an independent set A with $|A| > k$, contradicting $\alpha(G) \leq k$.
Hint use $A := \{v_{i+1} : i \in I\} \cup \{u\}$.

Solution:

Bonus exercises.

14: Give an example of a graph G that is
(a) Eulerian but not Hamiltonian.
(b) Hamiltonian but not Eulerian.
(c) Hamiltonian and has an Eulerian trail but is not Eulerian.
(d) neither Eulerian nor Hamiltonian, but has an Eulerian trail.

15: Is the following graph Hamiltonian?



16: Let G be a graph of order $n \geq 3$ such that $\deg u + \deg v \geq n-1$ for every two nonadjacent vertices u and v. Prove that G must contain a Hamiltonian path.

17: For $n \geq 2$, prove by induction on n that the maximum number of edges in a simple non-Hamiltonian n-vertex graph is $\binom{n-1}{2}+1$.

18: Let G be a graph that is not a forest and the shortest cycle in G has length at least 5. Prove that $\overline{G}$ is Hamiltonian.

A graph G is called t-tough, where $t > 0$ is any real number, if for every separator S the graph $G - S$ has at most $|S|/t$ components.

19: Show that every Hamiltonian graph is 1-tough.

20: Find a 1-tough graph that is not Hamiltonian.

Conjecture. (Chvátal 1973) There exists an integer t such that every t-tough graph has a Hamilton cycle.